Miscellaneous Examples

Example 25 If $\sin x = \frac{3}{5}$, $\cos y = -\frac{12}{13}$, where x and y both lie in second quadrant, find the value of $\sin(x + y)$.

Solution We know that

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \quad \text{... (1)}$$

Now $\cos^2 x = 1 - \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Therefore $\cos x = \pm\frac{4}{5}$.

Since x lies in second quadrant, $\cos x$ is negative.

Hence $\cos x = -\frac{4}{5}$

Now $\sin^2 y = 1 - \cos^2 y = 1 - \frac{144}{169} = \frac{25}{169}$

i.e. $\sin y = \pm\frac{5}{13}$.

Since y lies in second quadrant, hence $\sin y$ is positive. Therefore, $\sin y = \frac{5}{13}$. Substituting the values of $\sin x$, $\sin y$, $\cos x$ and $\cos y$ in (1), we get

$$\sin(x+y) = \frac{3}{5}\times\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\times\frac{5}{13} = -\frac{36}{65}-\frac{20}{65} = -\frac{56}{65}$$

Example 26 Prove that

$$\cos 2x \cos\frac{x}{2} - \cos 3x \cos\frac{9x}{2} = \sin 5x \sin\frac{5x}{2}.$$

Solution We have

$$\text{L.H.S.} = \frac{1}{2}\left[2\cos 2x\cos\frac{x}{2} - 2\cos\frac{9x}{2}\cos 3x\right]$$

$$= \frac{1}{2}\left[\cos\left(2x+\frac{x}{2}\right)+\cos\left(2x-\frac{x}{2}\right)-\cos\left(\frac{9x}{2}+3x\right)-\cos\left(\frac{9x}{2}-3x\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{5x}{2}+\cos\frac{3x}{2}-\cos\frac{15x}{2}-\cos\frac{3x}{2}\right] = \frac{1}{2}\left[\cos\frac{5x}{2}-\cos\frac{15x}{2}\right]$$

$$= \frac{1}{2}\left[-2\sin\left\{\frac{\frac{5x}{2}+\frac{15x}{2}}{2}\right\}\sin\left\{\frac{\frac{5x}{2}-\frac{15x}{2}}{2}\right\}\right]$$

$$= -\sin 5x \sin\left(-\frac{5x}{2}\right) = \sin 5x \sin\frac{5x}{2} = \text{R.H.S.}$$

Example 27 Find the value of $\tan\frac{\pi}{8}$.

Solution Let $x = \frac{\pi}{8}$. Then $2x = \frac{\pi}{4}$.

Now $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

or $\tan\frac{\pi}{4} = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

Let $y = \tan\frac{\pi}{8}$. Then $1 = \dfrac{2y}{1-y^2}$

or $$y^2 + 2y - 1 = 0$$

Therefore $$y = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

Since $\frac{\pi}{8}$ lies in the first quadrant, $y = \tan\frac{\pi}{8}$ is positve. Hence

$$\tan\frac{\pi}{8} = \sqrt{2} - 1.$$

Example 28 If $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$, find the value of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Solution Since $\pi < x < \frac{3\pi}{2}$, $\cos x$ is negative.

Also $$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}.$$

Therefore, $\sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative.

Now $$\sec^2 x = 1 + \tan^2 x = 1 + \frac{9}{16} = \frac{25}{16}$$

Therefore $$\cos^2 x = \frac{16}{25} \text{ or } \cos x = -\frac{4}{5} \quad \text{(Why?)}$$

Now $$2\sin^2\frac{x}{2} = 1 - \cos x = 1 + \frac{4}{5} = \frac{9}{5}.$$

Therefore $$\sin^2\frac{x}{2} = \frac{9}{10}$$

or $$\sin\frac{x}{2} = \frac{3}{\sqrt{10}} \quad \text{(Why?)}$$

Again $$2\cos^2\frac{x}{2} = 1 + \cos x = 1 - \frac{4}{5} = \frac{1}{5}$$

Therefore $$\cos^2\frac{x}{2} = \frac{1}{10}$$

or

$$\cos\frac{x}{2} = -\frac{1}{\sqrt{10}} \text{ (Why?)}$$

Hence

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3}{\sqrt{10}} \times \left(\frac{-\sqrt{10}}{1}\right) = -3.$$

Example 29 Prove that $\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right) = \frac{3}{2}$

Solution We have

$$\text{L.H.S.} = \frac{1+\cos 2x}{2} + \frac{1+\cos\left(2x+\frac{2\pi}{3}\right)}{2} + \frac{1+\cos\left(2x-\frac{2\pi}{3}\right)}{2}.$$

$$= \frac{1}{2}\left[3+\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\left(\pi-\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-2\cos 2x\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-\cos 2x\right] = \frac{3}{2} = \text{R.H.S.}$$

Exercise on Chapter 3

Summary

- If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\,\theta$
- Radian measure $= \dfrac{\pi}{180} \times$ Degree measure
- Degree measure $= \dfrac{180}{\pi} \times$ Radian measure
- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \operatorname{cosec}^2 x$
- $\cos(2n\pi + x) = \cos x$
- $\sin(2n\pi + x) = \sin x$
- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$

- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$ $\quad$ $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

  $\cos(\pi - x) = -\cos x$ $\quad$ $\sin(\pi - x) = \sin x$

  $\cos(\pi + x) = -\cos x$ $\quad$ $\sin(\pi + x) = -\sin x$

  $\cos(2\pi - x) = \cos x$ $\quad$ $\sin(2\pi - x) = -\sin x$

- If none of the angles x, y and $(x \pm y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

- $$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- If none of the angles x, y and $(x \pm y)$ is a multiple of π, then

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

- $$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$$

- $$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

- $\sin 2x = 2 \sin x \cos x = \dfrac{2 \tan x}{1 + \tan^2 x}$

- $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

- $\sin 3x = 3\sin x - 4\sin^3 x$

- $\cos 3x = 4\cos^3 x - 3\cos x$

- $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3\tan^2 x}$

- (i) $\cos x + \cos y = 2\cos \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

  (ii) $\cos x - \cos y = -2\sin \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

  (iii) $\sin x + \sin y = 2 \sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

  (iv) $\sin x - \sin y = 2\cos \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

- (i) $2\cos x \cos y = \cos (x + y) + \cos (x - y)$

  (ii) $-2\sin x \sin y = \cos (x + y) - \cos (x - y)$

  (iii) $2\sin x \cos y = \sin (x + y) + \sin (x - y)$

  (iv) $2 \cos x \sin y = \sin (x + y) - \sin (x - y)$.

- $\sin x = 0$ gives $x = n\pi$, where $n \in \mathbf{Z}$.

- $\cos x = 0$ gives $x = (2n + 1) \dfrac{\pi}{2}$, where $n \in \mathbf{Z}$.

- $\sin x = \sin y$ implies $x = n\pi + (-1)^n y$, where $n \in \mathbf{Z}$.

- $\cos x = \cos y$, implies $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$.

- $\tan x = \tan y$ implies $x = n\pi + y$, where $n \in \mathbf{Z}$.